REMARKS ON CRITICAL POINTS OF PHASE FUNCTIONS AND NORMS OF BETHE VECTORS

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ABSTRACT. We consider a tensor product of a Verma module and the linear representation of $sl(n+1)$. We prove that the corresponding phase function, which is used in the solutions of the KZ equation with values in the tensor product, has a unique critical point and show that the Hessian of the logarithm of the phase function at this critical point equals the Shapovalov norm of the corresponding Bethe vector.

1. INTRODUCTION

Let $\mathfrak{g}$ be a simple Lie algebra with simple roots α_i and Chevalley generators e_i, f_i, h_i, $i = 1, \dots, n$. Let V_1, V_2 be representations of $\mathfrak{g}$ with highest weights λ_1, λ_2. The Knizhnik-Zamolodchikov (KZ) equation on a function u with values in $V_1 \otimes V_2$ has the form

$$\kappa \frac{\partial}{\partial z_1} u = \frac{\Omega}{z_1 - z_2} u, \qquad \kappa \frac{\partial}{\partial z_2} u = \frac{\Omega}{z_2 - z_1} u,$$

where $\Omega \in \mathrm{End}(V_1 \otimes V_2)$ is the Casimir operator. Solutions with values in the space of singular vectors of weight $\lambda_1 + \lambda_2 - \sum_{i=1}^n l_j \alpha_j$ are given by hypergeometric integrals with $l = \sum_{i=1}^n l_j$ integrations, see [SV].

For an ordered set of numbers $I = \{i_1, \dots, i_m\}$, $i_k \in \{1, \dots, n\}$, and a vector v in a representation of $\mathfrak{g}$, denote $f^I v = f_{i_1} \dots f_{i_m} v$. The hypergeometric solutions of the KZ equation have the form

$$u = \sum u_{I,J} f^I v_1 \otimes f^J v_2, \qquad u_{I,J} = \int_\gamma \Omega \tilde{\omega}_{I,J} dt_1 \wedge \dots \wedge dt_l,$$

where v_1, v_2 are highest weight vectors of V_1, V_2; the summation is over all pairs of ordered sets I, J, such that their union $\{i_k, j_s\}$ contains a number i exactly l_i times, $i = 1, \dots, n$; γ is a suitable cycle; $\tilde{\omega}_{I,J} = \tilde{\omega}_{I,J}(z_1, z_2, t_1, ..., t_l)$ are suitable rational functions, the function $\Omega = \Omega(z_1, z_2, t_1, ..., t_l)$, called the phase function, is given by

$$\Omega = (z_1 - z_2)^{(\lambda_1, \lambda_2)/\kappa} \prod_{j=1}^{l} (t_j - z_1)^{-(\lambda_1, \alpha_{t_j})/\kappa} (t_j - z_2)^{-(\lambda_2, \alpha_{t_j})/\kappa} \prod_{1 \leqslant i < j \leqslant n} (t_i - t_j)^{(\alpha_{t_i}, \alpha_{t_j})/\kappa}.$$

Here $(\ ,\ )$ is the Killing form and α_{t_i} denotes the simple root assigned a the variable t_i by the following rule. The first l_1 variables $t_1, \dots, t_{l_1}$ are assigned to the simple root α_1, the next l_2 variables $t_{l_1+1}, \dots, t_{l_1+l_2}$ to the second simple root α_2, and so on.

Define the normalized phase function Φ by the formula

$$\Phi(\lambda_1, \lambda_2, \kappa) = \prod_{j=1}^{l} t_j^{-(\lambda_1, \alpha_{t_j})/\kappa} (1 - t_j)^{-(\lambda_2, \alpha_{t_j})/\kappa} \prod_{1 \leqslant i < j \leqslant n} (t_i - t_j)^{(\alpha_{t_i}, \alpha_{t_j})/\kappa}. \qquad (1)$$

We also substitute $z_1 = 0$, $z_2 = 1$ in the rational functions $\tilde{\omega}_{I,J}$ and denote the result $\omega_{I,J}$.

Conjecture 1. *If the space of singular vectors of weight $\lambda_1 + \lambda_2 - \sum_{i=1}^n l_j \alpha_j$ is one-dimensional, then there is a region Δ of the form $\Delta = \{t \in \mathbb{R}^l \mid 0 < t_{\sigma_l} < \cdots < t_{\sigma_1} < 1\}$ for some permutation σ, such that the integral $\int_\Delta \Phi dt$ can be computed explicitly and it is equal to an alternating product of Euler Γ-functions up to a rational number independent on $\lambda_1, \lambda_2, \kappa$.*

Example. *The Selberg integral.* Let $\mathfrak{g} = sl(2)$. Let V_1 and V_2 be $sl(2)$ modules with highest weights $\lambda_1, \lambda_2 \in \mathbb{C}$. Then the normalized phase function (1) has the form

$$\Phi(\lambda_1, \lambda_2, \kappa) = \prod_{j=1}^{l} t_j^{-\lambda_1/\kappa} (1 - t_j)^{-\lambda_2/\kappa} \prod_{1 \leqslant i < j \leqslant l} (t_i - t_j)^{2/\kappa}. \tag{2}$$

Conjecture 1 holds for $\mathfrak{g} = sl(2)$ according to the Selberg formula

$$l! \int_\Delta \Phi(\lambda_1, \lambda_2, \kappa) dt_1 \dots dt_l = \prod_{j=0}^{l-1} \frac{\Gamma((-\lambda_1 + j)/\kappa + 1)\Gamma((-\lambda_2 + j)/\kappa + 1)\Gamma((j+1)/\kappa + 1)}{\Gamma((-\lambda_1 - \lambda_2 + (2l - j - 2))/\kappa + 2)\Gamma(1/\kappa + 1)},$$

where $\Delta = \{t \in \mathbb{R}^l \mid 0 < t_1 < \cdots < t_l < 1\}$. $\square$

Using the phase function Φ and the rational functions $\omega_{I,J}$, one can construct singular vectors in $V_1 \otimes V_2$. Namely, if t^0 is a critical point of the function Φ, then the vector $\sum \omega_{I,J}(t^0) f^I v_1 \otimes f^J v_2$ is singular, see [RV]. The equation for critical points, $d\Phi = 0$, is called the *Bethe equation* and the corresponding singular vectors are called the *Bethe vectors.*

Conjecture 2. *If the space of singular vectors of a given weight in $V_1 \otimes V_2$ is one-dimensional, then the corresponding phase function has exactly one critical point modulo permutations of variables assigned to the same simple root.*

Example. The conjecture holds for $\mathfrak{g} = sl(2)$. If $(t_1, \dots, t_l)$ is a critical point of the function $\Phi(\lambda_1, \lambda_2, \kappa)$ given by (2), then

$$\sigma_k(t) = \binom{l}{k} \prod_{j=1}^{k} \frac{\lambda_1 - l + j}{\lambda_1 + \lambda_2 - 2l + j + 1},$$

where $\sigma_1(t) = \sum t_j$, $\sigma_2(t) = \sum t_i t_j$, etc, are the standard symmetric functions, see [V], so there is a unique critical point up to permutations of coordinates. $\square$

The rational functions $\omega_{I,J}(t)$ are invariant with respect to permutation of variables assigned to the same simple root. Thus, Conjecture 2 implies that there is a unique Bethe vector X.

The space $V_1 \otimes V_2$ has a natural bilinear form B, called the Shapovalov form, which is the tensor product of Shapovalov forms of factors.

Conjecture 3. *The length of a Bethe vector X equals the Hessian of the logarithm of the phase function Φ with $\kappa = 1$ at a critical point t^0,*

$$B(X, X) = det \left(\frac{\partial^2}{\partial t_i \partial t_j} \ln \Phi(t^0) \right).$$

Example. The conjecture holds for $\mathfrak{g} = sl(2)$, see [V]. $\square$

In this paper we prove Conjectures 1, 2 and 3 for the case when $\mathfrak{g} = sl(n+1)$, V_1 is a Verma module and V_2 is the linear representation.

2. The integral

Let

$$\tilde{\Phi}_n(\alpha,\beta) = t_1^{\alpha_1}(1-t_1)^{\beta_1}\prod_{j=2}^{n} t_j^{\alpha_j}(t_j-t_{j-1})^{\beta_j}. \tag{3}$$

Theorem 1. *Let $\alpha_i > 0$, $\beta_i > 0$, $i = 1, \dots n$. Then*

$$\int_{\Delta_n}\tilde{\Phi}_n(\alpha,\beta)dt_1\dots dt_n = \prod_{j=1}^{n}\frac{\Gamma(\beta_j+1)\Gamma(\alpha_j+\dots+\alpha_n+\beta_{j+1}+\dots+\beta_n+n-j+1)}{\Gamma(\alpha_j+\dots+\alpha_n+\beta_j+\dots+\beta_n+n-j+2)},$$

where $\Delta_n = \{t\in\mathbb{R}^n \mid 0 < t_n < \dots < t_1 < 1\}$.

Proof: The formula is clearly true for $n = 1$.

Fix $t_1, \dots, t_{n-1}$ and integrate with respect to t_n. We obtain the recurrent relation

$$\int_{\Delta_n}\tilde{\Phi}_n(\alpha,\beta)dt_1\dots dt_n = \frac{\Gamma(\alpha_n+1)\Gamma(\beta_n+1)}{\Gamma(\alpha_n+\beta_n+2)}\times$$
$$\times\int_{\Delta_{n-1}}\tilde{\Phi}_{n-1}(\alpha_1,\dots,\alpha_{n-1},\beta_1,\dots,\beta_{n-2},\beta_{n-1}+\beta_n+\alpha_n+1)dt_1\dots dt_{n-1},$$

which implies the Theorem. □

3. The critical point

Let $\mathfrak{g} = sl(n+1)$. Let V_1 be a Verma module of highest weight λ, $(\lambda,\alpha_i) = \lambda_i$. Let V_2 be the linear representation, that is the irreducible representation with highest weight ω, $(\omega,\alpha_i) = \delta_{i,1}$.

The nontrivial subspaces of singular vectors of a given weight in the tensor product $V_1\otimes V_2$ are one dimensional and have weights $\lambda+\omega-\sum_{i=1}^{k}\alpha_i$, $k = 0,\dots,n$. The computations for weights $\lambda+\omega-\sum_{i=1}^{k}\alpha_i$, $k < n$, are reduced to the case $\mathfrak{g} = sl(k+1)$. Consider the normalized phase function $\Phi_n(\lambda,\kappa)$ corresponding to the weight $\lambda+\omega-\sum_{i=1}^{n}\alpha_i$.

We have $\Phi_n(\lambda,\kappa) = \Phi(\lambda,\omega,\kappa)$, where $\Phi(\lambda,\omega,\kappa)$ is given by (1). Note that

$$\Phi_n(\lambda,\kappa) = \tilde{\Phi}_n(-\lambda_1/\kappa,\dots,-\lambda_n/\kappa,-1/\kappa,\dots,-1/\kappa),$$

where $\tilde{\Phi}_n$ is given by (3).

Theorem 2. *The function $\Phi_n(\lambda,\kappa)$ has exactly one critical point $t^n = (t_1^n,\dots,t_n^n)$ given by*

$$t_j^n(\lambda_1,\dots,\lambda_n) = \prod_{i=1}^{j}\frac{\lambda_i+\dots+\lambda_n+n-i}{\lambda_i+\dots+\lambda_n+n-i+1}.$$

Proof: The computation is obvious if $n = 1$.

The equation $\partial\Phi_n/\partial t_n = 0$ has the form

$$t_n^n = \frac{\lambda_n}{\lambda_n+1}t_{n-1}^n.$$

Substituting for t_n^n in the equations $\partial\Phi_n/\partial t_i = 0$, $i = 1,\dots,n-1$ and comparing the result with the equation $d\Phi_{n-1} = 0$, we obtain

$$t_k^n(\lambda_1,\dots,\lambda_n) = t_k^{n-1}(\lambda_1\dots,\lambda_{n-2},\lambda_{n-1}+\lambda_n+1), \qquad k = 1,\dots,n-1.$$

This recurrent relation implies the Theorem. □

4. The norm of the Bethe vector

Let V be a $\mathfrak{g}$ module with highest weight vector v. The Shapovalov form $B(\ ,\ ) : V \otimes V \to \mathbb{C}$ is the unique symmetric bilinear form with the properties

$$B(e_i x, y) = B(x, f_i y), \qquad B(v, v) = 1,$$

for any $x, y \in V$. The Shapovalov form on a tensor product of modules is the tensor product of Shapovalov forms of factors.

Let $\mathfrak{g} = sl(n+1)$. Let $V_1 = V_\lambda$ be a Verma module of highest weight λ. Let $V_2 = V_\omega$ be the linear representation. Then the space of singular vecors in $V_\lambda \otimes V_\omega$ of weight $\lambda + \omega - \sum_{i=1}^n \alpha_i$ is one-dimensional and is spanned by the Bethe vector $X^n(\lambda)$ corresponding to the critical point of the function $\Phi_n(\lambda, \kappa)$. The Bethe vector has the form

$$X^n(\lambda) = x_0^n \otimes f_n \dots f_1 v_0 + x_1^n \otimes f_{n-1} \dots f_1 v_0 + \dots + x_n^n \otimes v_0,$$

where $x_i^n \in V_\lambda$ and v_0 is the highest weight vector in V_ω. Here, $x_0^n = a^n v_\lambda$, where v_λ is the highest weight vector in V_λ and a^n is the value of the corresponding rational function

$$\omega_{\emptyset,(n,n-1,\dots,1)}(t) = \frac{1}{t_1 - 1} \prod_{i=1}^{n-1} \frac{1}{t_{i+1} - t_i}$$

at the critical point t_n of function $\Phi_n(\lambda, \kappa)$, given by Theorem 2. For a description of all other rational functions whose values at t^n determine $x_1^n, ..., x_n^n$, see [SV]. We have

$$a^n = (-1)^n \prod_{k=1}^n \frac{(\lambda_k + \dots + \lambda_n + n - k + 1)^{n-k+1}}{(\lambda_k + \dots + \lambda_n + n - k)^{n-k}}.$$

Theorem 3.

$$B(X^n(\lambda), X^n(\lambda)) = \prod_{k=1}^n \frac{(\lambda_k + \dots + \lambda_n + n - k + 1)^{2(n-k)+3}}{(\lambda_k + \dots + \lambda_n + n - k)^{2(n-k)+1}}. \tag{4}$$

Proof: We also claim

$$B(x_n^n, x_n^n) = \frac{B(X^n(\lambda), X^n(\lambda))}{\lambda_k + \dots + \lambda_n + n}. \tag{5}$$

Formulas (4), (5) are readily checked for $n = 1$.

The vectors $\{v_0, f_1 v_0, f_2 f_1 v_0, \ \dots, \ f_n \dots f_1 v_0\}$ form an orthonormal basis of V_ω with respect to its Shapovalov form. Clearly, we have

$$B(X^n(\lambda), X^n(\lambda)) = \left(\frac{a^n(\lambda)}{a^{n-1}(\lambda')} \right)^2 B(X^{n-1}(\lambda'), X^{n-1}(\lambda')) + B(x_n^n, x_n^n),$$

where λ' is the $sl(n)$ weight, such that $(\lambda', \alpha_i) = \lambda_{i+1}$, $i = 1, \dots, n-1$.

The vector X^n is singular. In particular it means that $e_i x_n^n = 0$ for $i > 1$ and $e_1 x_n^n = -x_{n-1}^n$. The vector x_n^n has the form $x_n^n = \sum_\sigma b_\sigma^n f_{\sigma(1)} \dots f_{\sigma(n)} v_\lambda^n$, where the coefficients b_σ^n are the values of the corresponding rational functions at the critical point given by Theorem 2.

Let $b^n = b_{\sigma=\mathrm{id}}^n$. Then we have

$$B(x_n^n, x_n^n) = B(x_n^n, b^n f_1 \dots f_n v_\lambda^n) = -b^n B(x_{n-1}^n, f_2 \dots f_n v_\lambda^n) =$$

$$= -b^n \frac{a_n}{a_{n-1}} B(x_{n-1}^{n-1}, f_1, \dots f_{n-1} v_{\lambda'}^{n-1}) = -\frac{b^n}{b^{n-1}} \frac{a_n}{a_{n-1}} B(x_{n-1}^{n-1}, x_{n-1}^{n-1}),$$

where x_{n-1}^{n-1} is a component of the singular vector in $V_{\lambda'} \otimes V_\omega$.

The coefficient b^n is the value of the function

$$\omega_{(n,n-1,\dots,1),\emptyset}(t) = \frac{1}{t_n} \prod_{i=1}^{n-1} \frac{1}{t_i - t_{i+1}}$$

at the critical point t^n, given by Theorem 2. We have

$$b^n = (-1)^{n-1} \frac{a_n}{\lambda_1 + \cdots + \lambda_n + n} \prod_{k=1}^{n} \frac{\lambda_k + \cdots + \lambda_n + n - k + 1}{\lambda_k + \cdots + \lambda_n + n - k}$$

Now, formulas (4), (5) are proved by induction on n. □

Theorem 4.

$$B(X^n(\lambda), X^n(\lambda)) = det \left(\frac{\partial^2}{\partial t_i \partial t_j} \ln \Phi_n(\lambda, \kappa = 1)(t^n) \right),$$

where t^n is the critical point of the phase function $\Phi_n(\lambda, \kappa)$ given by Theorem 2.

Proof: It is sufficient to prove the Theorem for $\lambda_i > 0, \kappa < 0$. We tend κ to zero and compute the asymptotics of the integral $\int_{\Delta_n} \Phi_n dt$.

On one hand, the integral is evaluated by Theorem 1. We compute the asymptotics using the Stirling formula for Γ-functions.

On the other hand, the asymptotics of the same integral can be computed by the method of stationary phase, since the critical point t^n of the function Φ_n is non-degenerate by Theorem 1.2.1 in [V]. Then the asymptotics of the integral is

$$(2\pi\kappa)^{l/2} \Phi_n(\lambda, \kappa)(t^n) \left(\mathrm{Hess}(\kappa \ln \Phi_n(\lambda, \kappa)(t^n)\right)^{-1/2}.$$

Note that $\kappa \ln \Phi_n(\lambda, \kappa) = \ln \Phi_n(\lambda, 1)$, and

$$\Phi_n(\lambda, \kappa)(t^n) = \prod_{k=1}^{n} \frac{(\lambda_k + \cdots + \lambda_n + n - k + 1)^{(\lambda_k + \cdots + \lambda_n + n - k + 1)/\kappa}}{(\lambda_k + \cdots + \lambda_n + n - k)^{(\lambda_k + \cdots + \lambda_n + n - k)/\kappa}}.$$

Comparing the results we compute the Hessian explicitly and prove the Theorem. □

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